Exercise 6

Solve the differential equation.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x^2$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2y_c}{dx^2} + \frac{dy_c}{dx} - 2y_c = 0 \tag{1}$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad \frac{dy_c}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y_c}{dx^2} = r^2e^{rx}$$

Substitute these formulas into the ODE.

$$r^2e^{rx} + re^{rx} - 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + r - 2 = 0$$

Solve for r.

$$(r+2)(r-1) = 0$$

$$r = \{-2, 1\}$$

Two solutions to the ODE are e^{-2x} and e^x. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$y_c(x) = C_1e^{-2x} + C_2e^x$$

C_1 and C_2 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2y_p}{dx^2} + \frac{dy_p}{dx} - 2y_p = x^2 \tag{3}$$

Since the inhomogeneous term is a polynomial of degree 2, the particular solution is $y_p = Ax^2 + Bx + C$.

$$y_p = Ax^2 + Bx + C \quad \rightarrow \quad \frac{dy_p}{dx} = 2Ax + B \quad \rightarrow \quad \frac{d^2y_p}{dx^2} = 2A$$

Substitute these formulas into equation (3).

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

$$(2A + B - 2C) + (2A - 2B)x + (-2A)x^2 = x^2$$

Match the coefficients to get a system of equations for A, B, and C.

$$\begin{aligned} 2A + B - 2C &= 0 \\ 2A - 2B &= 0 \\ -2A &= 1 \end{aligned}$$

Solving it yields

$$A = -\frac{1}{2} \quad \text{and} \quad B = -\frac{1}{2} \quad \text{and} \quad C = -\frac{3}{4}.$$

The particular solution is then

$$y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}.$$

Therefore, the general solution to the original ODE is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^{-2x} + C_2 e^{x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}. \end{aligned}$$